THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 11 Homotopy

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $\mathcal{M}$ be the set of all $n \times n$ real matrices. Any matrix $f \in \mathcal{M}$ can be seen as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.
   (a) Show that any $f, g \in \mathcal{M}$ are homotopic.
   (b) Is the homotopy between f, g above only involves mappings in $\mathcal{M}$? That is, there exists a homotopy $H\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ between f, g such that for each $t \in [0,1]$, the mapping $x \mapsto H(x,t)$ also belongs to $\mathcal{M}$. We call it a homotopy through mappings in $\mathcal{M}$.
   (c) Let $\mathcal{A} \subset \mathcal{M}$ be the subset of invertible matrices and $f, g \in \mathcal{A}$. Are they homotopic through mappings in $\mathcal{A}$?
   (d) If $f, g \in \mathcal{P}$, the set of positive definite matrices, then there is a homotopy between f and g through mappings in $\mathcal{P}$.
2. Let $\mathcal{M}$ be the set of all $n \times n$ real matrices. It can be given a topology induced by the standard $\mathbb{R}^{n^2}$. Show that $\mathcal{M}$ is path connected if and only if every pair of $f, g \in \mathcal{M}$ are homotopic through mappings in $\mathcal{M}$.
3. If $f_1 \simeq g_1\colon X \to Y_1$ and $f_2 \simeq g_2\colon X \to Y_2$, show that $(f_1, f_2) \simeq (g_1, g_2)$ as mappings $X \to (Y_1 \times Y_2)$, where $(f_1, f_2)(x) = (f_1(x), f_2(x))$ and $(g_1, g_2)(x) = (g_1(x), g_2(x))$.
4. Let $\mathbb{D} = \{\, z \in \mathbb{C}\colon\ |z| \le 1 \,\}$ and $f\colon \mathbb{D} \to \mathbb{D}$ be given by $f(z) = ze^{2\pi|z|\mathbf{i}}$. Show that f is homotopic to the identity mapping on $\mathbb{D}$. Geometrically visualize the action.

   Note that f is indeed a homeomorphism. Can you find a homotopy H such that at every $t \in [0,1]$, the map $z \mapsto H(z,t)$ is also a homeomorphism on $\mathbb{D}$?
5. Let $A = \{\, z \in \mathbb{C}\colon\ 1 \le |z| \le 2 \,\}$ and $f\colon \mathbb{D} \to \mathbb{D}$ be a homeomorphism on A given by $f(z) = ze^{2\pi(|z|-1)\mathbf{i}}$. Is it homotopy to the identity mapping on A?
6. Let $f, g\colon X \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be two mappings such that for all $x \in X$, $f(x) \ne -g(x)$. Show that $(1-t)f(x) + tg(x)$ will give a homotopy between f and g.
7. Show that if $f\colon X \to \mathbb{S}^n$ is not surjective, then f is null homotopic.

8. Let $a\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the antipodal map, i.e., $a(z) = -z$. Show that $a \simeq \mathrm{id}_{\mathbb{S}^1}$. Note that this is not true for $\mathbb{S}^n$ with even n but it holds for odd n.

9. Let Y be any topological space. Form the quotient space $CY = (Y \times [0,1])/\sim$ by the equivalence relation $\sim$ on $Y \times [0,1]$ with $(y_1, t_1) \sim (y_2, t_2)$ if $t_1 = 1 = t_2$. That is, CY is obtained by crushing the "top" $Y \times \{1\}$ to one point. Prove that any map $f\colon X \to CY$ is null homotopic.

10. Show that a map $f\colon X \to Y$ is null homotopic if and only if there exists $\tilde{f}\colon CX \to Y$ such that $\tilde{f}|_X \equiv f$ by naturally seeing $X \hookrightarrow CX$ as a subspace.

11. Show that homotopy equivalence (homotopy type) defines an equivalence relation on all the topological spaces.

12. Show that a space of two points, i.e., $X = \{-1, 1\}$ with discrete topology, is not homotopy equivalent to a one point space. In other words, X is not contractible.

13. Try to convince yourself that $\mathbb{S}^n$ is not contractible (the rigorous proof may be beyond your knowledge now, see exercise below).

    Remark. Note that the two-point space above is defined as the 0-dimensional sphere, $\mathbb{S}^0$.

14. Consider the unit sphere $\mathbb{S}^{n-1}$ and the punctured space $\mathbb{R}^n \setminus \{0\}$. Show that they are homotopy equivalent. In fact, $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

15. Give explicit argument of why $\mathbb{S}^1 \vee \mathbb{S}^1$ is a deformation retract of a one-punctured torus.

16. Given $f\colon X \to Y$, there is a natural mapping, again denote it by f, from $X \times \{0\} \to Y$. One may define the quotient spaces (called mapping cylinder and mapping cone),

$$\begin{aligned} M_f &= ((X \times [0,1]) \coprod Y)/\sim, \quad \text{where } (x,0) \sim f(x); \\ C_f &= ((X \times [0,1]) \coprod Y)/\sim, \quad \text{where } (x,0) \sim f(x) \text{ and } (x_1,1) \sim (x_2,1). \end{aligned}$$

    Remark. To understand them, imagine $f\colon \mathbb{S}^1 \to \mathbb{R}$ to be the standard embedding. Then M_f is a tall hat while C_f is a wizard hat. In general, f need not to be one-to-one. In addition, if $\mathbb{D}^n$ is the closed n-dimensional unit disk and $f\colon \mathbb{S}^n \to \mathbb{D}^{n+1}$ is the standard embedding, then $C_f = \mathbb{S}^{n+1}$.

    Show that if $f, g\colon X \to Y$ are homotopic mappings, then M_f and M_g are homotopy equivalent; likewise, C_f and C_g are also homotopy equivalent.

    Remark. Using this, one may prove the USELESS result: if $\mathbb{S}^n$ is contractible then so is $\mathbb{S}^{n+1}$. The converse is the USEFUL part because one may set up an induction process. Together with that $\mathbb{S}^0$ is not contractible (done above), we prove $\mathbb{S}^n$ is not contractible.

    For those who are interested, you may try to show if both X and Y are Hausdorff, then so are M_f and C_f.